

Static consensus in passifiable linear networks

Ibragim A. Junussov *

March 3, 2014

Abstract

Sufficient conditions of consensus in networks described by digraphs and consisting of identical SIMO systems are derived. It is assumed that each system is passifiable. Cases of identical and nonidentical control gains are considered. Scalability for circle digraph in terms of gain magnitudes is studied.

1 Introduction

Control of multi-agent systems has attracted significant interest in last decade since it has a great technical importance [4, 14, 17] and relates to biological systems [18].

In consensus problems agents communicate via decentralized controllers using relative measurements with a final goal to achieve common behaviour (synchronization) which can evolve in time. Many approaches have been developed for a different problem settings.

In [22] performance in undirected graphs are studied. In [21] adaptive coupling strengths are considered. Communication between all agents except leader (if its presented) assumed to be bidirectional.

Conception of synchronization region in complex plane for a networks consisting of linear dynamical systems is introduced in [12]. In [23] this conception is used for analysis of synchronization with leader. Problem is solved using Linear Quadratic Regulator approach in cases when full state is available for measurement and when its not. In last case observers are constructed.

Laplace matrix and its spectrum plays crucial role in description and analysis of consensus problems. It has broad applications, e.g. [10]. Conditions which are causing unit multiplicity of digraph Laplace matrix zero eigenvalue have been found in [1]. Analysis of tree structure and Laplace matrices spectrum of digraphs are also studied by these authors. Work [5] contains examples of out-forests as well as useful graph theoretical concepts and can be recommended as an entry reading to the research of these authors on algebraic digraph theory and consensus problems.

Scalability is one of the questions which should be considered. Control of platoons – one dimensional formations of automated vehicles, is a field related to consensus problems. It is known that in platoons string instability effect (amplified response of trailing vehicles to a leader disturbance while platoon length is growing) is taking place [19, 20, 16]. This effect

*I.A. Junussov is with Department of Control Engineering, Czech Technical University in Prague, e-mail: dxdtfxut@gmail.com The work was supported by the Ministry of Education, Youth and Sports of the Czech Republic within the project no. CZ.1.07/2.3.00/30.0034 (Support for improving R&D teams and the development of intersectoral mobility at CTU in Prague). Author would like to thank A.L. Fradkov, Z. Hurak, A. Selivanov and I. Herman for useful discussions.

can be avoided by adding certain level of centralization [16] or, possibly, by considering more sophisticated communication strategy and nonlinear control.

In this paper results of Passification Method [8, 9] are used to synthesize a decentralized control law and to derive sufficient conditions of full state synchronization by relative outputs in a networks described by digraphs with dynamical nodes modelled as linear systems. Assumptions made on network topology are minimal, both leader and leaderless cases are treated.

Paper is organized as follows. In Section 2.1 some preliminaries are given. Problem statement and assumptions are given in Sect. 2.2. Sufficient conditions of consensus with identical gains are given in Sect. 2.3, Theorem 1. Assumptions of Theorem 1 are relaxed in comparison with Theorem 2 from [7]. Sufficient conditions of consensus with nonidentical gains and geometrical interpretation are given in Sect. 2.4. It is determined that boundary of sufficient gain region is a hypersurface in according gain space. Adaptive controller is proposed without rigour validation in Sect. 2.5. Theoretical results are illustrated by numerical simulations in networks consisting of double integrators. Scalability in a circle digraphs in terms of gain magnitudes is considered in Sect. 3.2. Numerical simulations show that sufficient lower bound on identical gains is approaching true bound as number of agents growing; identical gains in large circle digraphs should grow quadratically in number of agents. Results of Sect. 2.4 are illustrated by three-node digraph in Sect. 3.3. Control with nonidentical gains for this digraph demonstrates performance which is not worse than performance with identical gains.

2 Theoretical study

2.1 Preliminaries and notations

In this section notations, some terms of graph theory and Passification Lemma are listed.

2.1.1 Notations

Notation $\|\cdot\|_2$ stands for Euclidian norm. For two symmetric matrices M_1, M_2 inequality $M_1 > M_2$ means that matrix $M_1 - M_2$ is positive definite. Notation $\text{col}(v_1, \dots, v_d)$ stands for vector $(v_1, \dots, v_d)^T$. Identity matrix of size d is denoted by I_d . Vector $\mathbf{1}_d = (1, 1, \dots, 1)$ is vector of size d and consisting of ones. Vector $\mathbf{0}_d$ is defined similarly. Matrix $\text{diag}(v_1, \dots, v_d)$ is square matrix whose i -th element on main diagonal is $v_i, i = 1, \dots, d$; other entries are zeroes. Notation \otimes stands for Kronecker product of matrices. Definition and properties of Kronecker product, including eigenvalues property, can be found in [2, 13].

2.1.2 Terms of graph theory

A pair $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where \mathcal{V} – set of vertices, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ – set of arcs (ordered pairs), is called digraph (directed graph). Let \mathcal{V} have N elements, $N \in \mathbb{N}$. It is assumed hereafter that graphs does not have a self-loops, i.e. for any vertex $\alpha \in \mathcal{V}$ arc $(\alpha, \alpha) \notin \mathcal{E}$.

Digraph is called directed tree if all it vertices except one (called root) have exactly one parent. Let us agree that in any arc $(\alpha, \beta) \in \mathcal{E}$ vertex β is parent or neighbour. Directed spanning tree of a digraph \mathcal{G} is a directed tree formed of all digraph \mathcal{G} vertices and some of its arcs such that there exists path from any vertex to the root vertex in this tree.

A digraph is called weighted if to any pair of vertices $\alpha, \beta \in \mathcal{E}$ number $w(\alpha, \beta) \geq 0$ is assigned such that:

$$w(\alpha, \beta) > 0 \quad \text{if} \quad (\alpha, \beta) \in \mathcal{E} \quad \text{and} \quad w(\alpha, \beta) = 0 \quad \text{if} \quad (\alpha, \beta) \notin \mathcal{E}.$$

A digraph in which all nonzero weights are equal to 1 will be referred as unit weighted.

An adjacency matrix $\mathcal{A}(\mathcal{G})$ is $N \times N$ matrix whose i -th, j -th entry is equal to $w(\alpha_i, \alpha_j)$, $i, j = 1, \dots, N$.

Laplace matrix of digraph \mathcal{G} is defined as follows:

$$L(\mathcal{G}) = \text{diag}(\mathcal{A}(\mathcal{G}) \cdot \mathbf{1}_N) - \mathcal{A}(\mathcal{G}).$$

Matrix $L(\mathcal{G})$ always has zero eigenvalue with corresponding right eigenvector $\mathbf{1}_N : L(\mathcal{G}) \cdot \mathbf{1}_N = 0 \cdot \mathbf{1}_N$. By construction and Gershgorin Circle Theorem all eigenvalues of L have nonnegative real parts.

2.1.3 Passification Lemma

Problem of linear system passification is a problem of finding static linear feedback which is making initial system passive. It was solved in [8, 9] for nonsquare SIMO and MIMO systems including case of complex parameters. Brief outline of SIMO systems passification is given below.

Let A, B, C be real matrices of sizes $n \times n, n \times 1, n \times l$ accordingly. Denote by $\chi(s) = C^T(sI - A)^{-1}B$, $s \in \mathbb{C}$. Let vector $g \in \mathbb{R}^l$. If numerator of function $g^T \chi(s)$ is Hurwitz with degree $n - 1$ and has positive coefficients then function $g^T \chi(s)$ is called hyper-minimum-phase.

Lemma 1 (*Passification Lemma [8, 9]*) *If there exists vector $g \in \mathbb{R}^l$ such that function $g^T \chi(s)$ is hyper-minimum-phase, then following is true. There exists number $\varkappa_0 > 0$ such that for any $\varkappa > \varkappa_0$ there exists $n \times n$ real matrix $H = H^T > 0$ satisfying following matrix relations*

$$HA_* + A_*^T H < 0, \quad HB = Cg, \quad A_* = A - \varkappa Bg^T C^T. \quad (1)$$

2.2 Problem statement and assumptions

Consider a network consisting of N agents modelled as linear dynamical systems:

$$\begin{aligned} \dot{x}_i(t) &= Ax_i(t) + Bu_i(t), \\ y_i(t) &= C^T x_i(t), \end{aligned} \quad (2)$$

where $i = 1, \dots, N$, $x_i \in \mathbb{R}^n$ – state vector, $y_i \in \mathbb{R}^l$ – output or measurements vector, $u_i \in \mathbb{R}^1$ – input or control, A, B, C are real matrices of according size. By associating agents with N vertices of unit weighted digraph \mathcal{G} and introducing set of arcs one can describe information flow in the network. For $i = 1, \dots, N$ let us introduce notation for relative outputs

$$\bar{y}_i(t) = \sum_{j \in \mathcal{N}_i} (y_i(t) - y_j(t)),$$

where \mathcal{N}_i is a set of i -th agents neighbours.

Problem is to design controllers which use relative outputs and ensure achievement of the state synchronization (consensus) of all agents:

$$\lim_{t \rightarrow \infty} (x_i(t) - x_j(t)) = 0, \quad i, j = 1, \dots, N. \quad (3)$$

In the case of synchronization achievement asymptotical behaviour of all agents will be described by same time-dependant consensus vector which is denoted hereafter by $c(t)$:

$$\lim_{t \rightarrow \infty} (x_i(t) - c(t)) = 0, \quad i = 1, \dots, N.$$

Let us make following assumption about dynamics of a single agent.

A1) *There exists vector $g \in \mathbb{R}^l$ such that function $g^T \chi(s) = g^T C^T (sI_n - A)^{-1} B$ is hyper-minimum-phase.*

Now let us make an assumption on graph topology.

A2) *Digraph \mathcal{G} has a directed spanning tree.*

Zero eigenvalue of Laplace matrix L has unit multiplicity iff this assumption holds [1].

2.3 Static identical control

Denote $r(L) = \min_{\lambda_i \neq 0} \operatorname{Re} \lambda_i$ where λ_i are eigenvalues of L . Under assumption A2 zero eigenvalue is simple. By properties of L other eigenvalues lie in open right half of complex plane, so $r(L)$ is positive number.

Suppose that assumption A1 holds with known vector $g \in \mathbb{R}^l$. Consider following static consensus control with identical gain $k_s \in \mathbb{R}^1, k_s > 0$:

$$u_i(t) = -k_s g^T \bar{y}_i(t), \quad i = 1, \dots, N, \quad (4)$$

where relative output $\bar{y}_i(t)$ has been defined in previous section. Denote by $v(L) \in \mathbb{R}^N$ left eigenvector of L which is corresponding to zero eigenvalue and scaled such that $v(L)^T \cdot \mathbf{1}_N = 1$ where $\mathbf{1}_N = \operatorname{col}(1, 1, \dots, 1) \in \mathbb{R}^N$. Denote $x(t) = \operatorname{col}(x_1(t), \dots, x_N(t))$.

Theorem 1 *Let assumptions A1 and A2 hold. Then for all k_s such that*

$$k_s > \frac{\varkappa_0}{r(L)} \quad (5)$$

controller (4) ensures achievement of goal (3) in dynamical network (2); asymptotical behaviour is described by following consensus vector

$$c(t) = \exp(At)(v(L)^T \otimes I_n)x(0).$$

Proof. Closed loop system (2), (4) can be rewritten in a following form

$$\dot{x}(t) = (I_N \otimes A - k_s L \otimes B g^T C^T)x(t). \quad (6)$$

Consider nonsingular real matrix P such that

$$\Lambda = \begin{pmatrix} 0 & \mathbf{0}_N^T \\ \mathbf{0}_N & \Lambda_e \end{pmatrix} = P^{-1}LP,$$

where $\Lambda_e \in \mathbb{R}^{(N-1) \times (N-1)}$. All eigenvalues of Λ_e have positive real parts. By considering first (zero) columns of matrices $P\Lambda = LP$ and $(P^T)^{-1}\Lambda^T = L^T(P^{-1})^T$ we can accept that first column of P is $\mathbf{1}_N$ and first row of P^{-1} is $v(L)^T$.

Let us apply coordinate transformation $z(t) = (P^{-1} \otimes I_n)x(t)$ and rewrite (6):

$$\dot{z}_1(t) = Az_1(t), \quad (7)$$

$$\dot{z}_e(t) = ((I_{N-1} \otimes A) - k_s(\Lambda_e \otimes B g^T C^T))z_e(t), \quad (8)$$

where $z_1 \in \mathbb{R}^n, z_e \in \mathbb{R}^{(N-1)n}, z = \operatorname{col}(z_1, z_e)$. If zero solution of (8) is globally asymptotically stable, then the statements of theorem are true.

For any fixed k_s satisfying (5) there exists $0 < \varepsilon_s < 1$ such that

$$\varepsilon_s k_s > \frac{\varkappa_0}{r(L)}.$$

Eigenvalues of matrix $(\Lambda_e - \varepsilon_s r(L)I_{N-1})$ have positive real parts. Therefore [3] there exists $(N-1) \times (N-1)$ real matrix $Q = Q^T > 0$ such that following Lyapunov inequality holds

$$(\Lambda_e - \varepsilon_s r(L)I_{N-1})^T Q + Q(\Lambda_e - \varepsilon_s r(L)I_{N-1}) > 0.$$

We can rewrite last inequality

$$\Lambda_e^T Q + Q\Lambda_e > 2\varepsilon_s r(L)Q.$$

By assumption A1 there exists $H = H^T > 0$ such that (1) is true with $\varkappa = \varepsilon_s k_s r(L)$, since $\varkappa > \varkappa_0$. Considering following Lyapunov function

$$V(z_e(t)) = z_e^T(t)(Q \otimes H)z_e(t)$$

and taking its time derivative along the nonzero trajectories of (8), we obtain

$$\begin{aligned} \frac{d}{dt}V(z_e(t)) &= z_e^T(t)(Q \otimes (A^T H + H A) - k_s(\Lambda_e^T Q + Q \Lambda_e) \otimes (C g g^T C^T))z_e(t) \leq \\ &\leq z_e^T(t)(Q \otimes (A^T H + H A) - 2k_s \varepsilon_s r(L) Q \otimes (C g g^T C^T))z_e(t) = \\ &= z_e^T(t)(Q \otimes ((A^T - \varkappa C g B^T)H + H(A - \varkappa B g^T C^T)))z_e(t) = \\ &= z_e^T(t)(Q \otimes (A_*^T H + H A_*))z_e(t) < 0. \end{aligned}$$

Matrix relations (1) have been used here. Last inequality concludes proof.

2.4 Nonidentical control and Gain Region

Let the initial digraph \mathcal{G} be unit weighted. Let us fix Laplace matrix L and consider static control with nonidentical gains $k_i > 0$:

$$u_i(t) = -k_i g^T \bar{y}_i(t), \quad i = 1, \dots, N. \quad (9)$$

Denote $k'_i = \frac{k_i}{k_s}, k_s > 0$ where point $k' = (k'_1, \dots, k'_N) \in \mathbb{R}^N$ lies on a part of unit sphere: $\sum_{i=1}^N (k'_i)^2 = 1$. This part lies in \mathbb{R}^N orthant

$$\mathcal{O} = \{(k_1, \dots, k_N) \in \mathbb{R}^N | k_i > 0, i = 1, \dots, N\}.$$

Without loss of generality we can assume that network does not have a leader, since in leader case we can reduce following consideration of synchronization gain region to lower dimension $N - 1$.

Denote by $K' = \text{diag}(k'_1, k'_2, \dots, k'_N)$. Laplace matrices L and $K'L$ correspond to same digraphs which differ only in weights of arcs (recall that $k_i > 0, i = 1, \dots, N$). Equation for closed loop system (2),(9) can be rewritten as follows

$$\dot{x}(t) = (I_N \otimes A - k_s(K'L \otimes B g^T C^T))x(t).$$

By repeating proof of Theorem 1 we can formulate following result.

Theorem 2 *Let assumptions A1 and A2 hold. Then for all $k_i = k_s \cdot k'_i$ such that*

$$\sum_{i=1}^N (k'_i)^2 = 1, \quad k_s > \frac{\varkappa_0}{r(K'L)}$$

controller (9) ensures achievement of goal (3) in dynamical network (2); asymptotical behaviour is described by following consensus vector

$$c(t) = \exp(At)(v(K'L)^T \otimes I_n)x(0). \quad (10)$$

Denote by $\mathcal{K} \subset \mathcal{O}$ region in orthant such that for any $(k_1, \dots, k_N) \in \mathcal{K}$ control (9) ensures achievement of the goal (3) in network (2), (9). Consider following region

$$\mathcal{K}_r = \left\{ (k_1, \dots, k_N) \in \mathcal{O} \left| k_i = k_s \cdot k'_i, \sum_{1 \leq i \leq N} (k'_i)^2 = 1, k_s > \frac{\varkappa_0}{r(K'L)} \right. \right\},$$

which is subset of $\mathcal{K} : \mathcal{K}_r \subset \mathcal{K}$. Let us denote

$$S'_\varepsilon = \left\{ (k'_1, \dots, k'_N) \left| k'_i \geq \varepsilon, i = 1, \dots, N, \sum_{1 \leq i \leq N} (k'_i)^2 = 1 \right. \right\}, \quad \varepsilon > 0.$$

Point on S'_ε determines ray (half-line) in \mathcal{O} with initial point at the origin. According to Theorem 2, by moving along this ray from origin, i.e. increasing k_s , we will reach \mathcal{K}_r . Consider map

$$h : k' \rightarrow \frac{\varkappa_0}{r(K'L)} \cdot k', \quad k' \in S'_\varepsilon,$$

which is continuous as a composition of continuous maps [11]. Image of this map is a subset of boundary $\partial\mathcal{K}_r$, therefore, by continuity of map h , boundary $\partial\mathcal{K}_r$ is a hypersurface in \mathbb{R}^N . Domain S'_ε is compact, so we can apply Weierstrass Extreme Value Theorem and arrive at following lemma.

Lemma 2 *Map $h : S'_\varepsilon \rightarrow h(S'_\varepsilon) \subset \partial\mathcal{K}_r$ is continuous and has minimum and maximum.*

Generally, hypersurface $\partial\mathcal{K}_r$ is not smooth in all its points.

2.5 Adaptive control

Consider the following adaptive controller:

$$\begin{aligned} u_i(t) &= -k_i(t)g^T \bar{y}_i(t), \\ \frac{d}{dt}k_i(t) &= \bar{y}_i(t)^T g g^T \bar{y}_i(t), \end{aligned} \tag{11}$$

Nonrigorously, when adaptive controller (11) is applied all $k_i(t)$ will grow until they reach \mathcal{K} .

Conjecture 1 *Under assumptions A1, A2 adaptive controller (11) ensures achievement of goal (3) with nonzero (for almost all initial conditions) consensus vector in closed loop system (2),(11).*

3 Examples and numerical simulations results

3.1 Agents description

Suppose that each agent S_i in a network is modelled as follows

$$\begin{aligned} \dot{x}_i &= Ax_i + Bu_i, \quad y_i = C^T x_i, \quad i = 1, \dots, N, \\ A &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}. \end{aligned} \tag{12}$$

For $g = 1$ transfer function $g^T \chi(s) = C^T (sI_2 - A)^{-1} B = \frac{s+1}{s^2}$ is hyper-minimum-phase. It can be shown that number $\varkappa_0 = 1$. In the next sections different digraphs describing network topology will be considered.

3.2 Cycle digraph and scalability

Denote by L_N^C Laplace matrix which is corresponding to unit weighted cycle digraph which is consisting of N nodes S_i with the set of arcs

$$(S_1, S_2) \cup \dots \cup (S_j, S_{j+1}) \cup \dots \cup (S_{N-1}, S_N) \cup (S_N, S_1).$$

Eigenvalues of L_N^C are evenly located at circle in complex plane [6]:

$$\lambda_k = 1 - \exp\left(\sqrt{-1} \cdot k \cdot \frac{2\pi}{N}\right), k = 0, \dots, N-1.$$

So, number $r(L_N^C)$ from (5) is equal to $1 - \cos \frac{2\pi}{N}$. For a large increasing number of agents N gain k_s should grow as N^2 :

$$\inf k_s \sim \frac{\varkappa_0}{2\pi^2} \cdot N^2, \quad N \rightarrow \infty, \quad (13)$$

where $\inf k_s = \varkappa_0 / r(L_N^C)$ is bound given by Theorem 1. Relation (13) can be obtained using Taylor series.

Denote by ρ_N ratio of gain synchronization bound $k_{sim}(N)$ obtained by numerical simulations to bound given by Theorem 1:

$$\rho_N = k_{sim}(N) : \left(\frac{\varkappa_0}{r(L_N^C)}\right).$$

In the next table approximate values of ρ_N for some N are given. Simulations was performed with dynamic nodes described in section 3.1 and identical gains.

N	4	5	6	7	8	9	10	20
ρ_N	0.5	0.66	0.75	0.82	0.87	0.88	0.91	0.97

If a ratio of true gain synchronization bound to $\inf k_s$ is near to 1 (or nondecreasing), while number of agents is large and growing, then it is possible to conclude that consensus in large cycle digraphs is hard to achieve, since an arbitrary high gains are not physically realizable, see (13).

On other hand, it is worth noting that cycle digraph is the graph with smallest number of edges which is delivering average consensus among all its nodes.

3.3 Small digraph and gain region

Consider digraph shown on Fig. 1 with dynamic nodes described in section 3.1. By Lemma 2 distance from origin to $\partial\mathcal{K}_r$ reaches minimum. Boundary $\partial\mathcal{K}_r$ for considering case is presented on Fig. 2. By variating eigenvalues of matrix $K'L$ we can state that minimum is realized on a point for which $k_2 : k_3 = 2 : 1$.

Let us compare performance in two cases:

- Identical gains $k_2^{(1)} = k_3^{(1)} = 0.527 \cdot k$;
- Nonidentical gains $k_2^{(2)} = \frac{2}{3} \cdot k$, $k_3^{(2)} = \frac{1}{3} \cdot k$.

Note that $\left\| \begin{pmatrix} k_2^{(1)} \\ k_3^{(1)} \end{pmatrix} \right\|_2 \approx \left\| \begin{pmatrix} k_2^{(2)} \\ k_3^{(2)} \end{pmatrix} \right\|_2$. By Theorem 2 factor k is as follows

$$k = \frac{3}{2} = \frac{\varkappa_0}{r(K^{(2)}L)}, \quad K^{(2)} = \begin{pmatrix} k_2^{(2)} & 0 \\ 0 & k_3^{(2)} \end{pmatrix}.$$

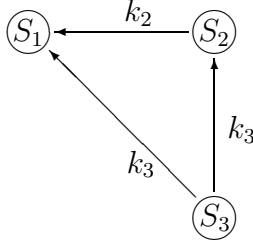


Figure 1: Digraph of 3 nodes

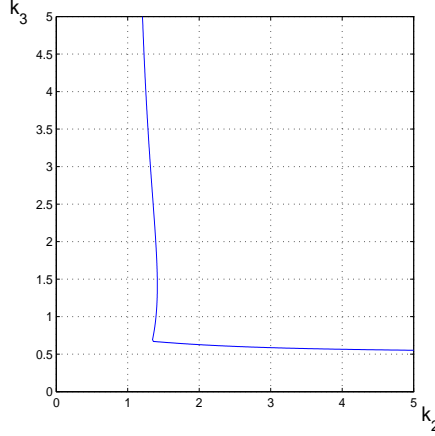


Figure 2: Gain region boundary $\partial\mathcal{K}_r$.

Let us choose following initial conditions:

$$x_1(0) = \text{col}(2, -2), x_2(0) = \text{col}(-7, 3), x_3(0) = \text{col}(1, -3).$$

Denote by $e(t) = \sum_{i=1}^2 \|x_i(t) - x_{i+1}(t)\|_2$ sum of error norms or disagreement measure; $e^{(1)}(t)$ error in the first case, $e^{(2)}(t)$ error in the second case. Results of 25 sec. simulations are shown on Fig. 3. From this figure one can observe following:

1. Control with nonidentical gains demonstrates better performance on initial period.
2. Overall synchronization time is almost same in both cases.

First observation does not hold generally for simulations with other initial conditions; second does hold. Note that consensus vector (10) does not changes for all $(k_2, k_3) \in \mathcal{K}$ since subsystem S_1 is leader.

4 Conclusions

By means of Passification Method [9] sufficient conditions of consensus with identical and non-identical gains are derived. Synchronous behaviour (consensus vector) is described, it can be affected by nonodentical gains in leaderless case. In [15] it is stated that there is tradeoff between performance and communication cost. Growth of gain magnitude in growing circle digraphs which have lowest communication cost for reaching average consensus is studied. For other graph topologies gain magnitude should grow slower. Three node digraph with a leader is studied: control with nonidentical gains provide performance which is not worse than identical control.

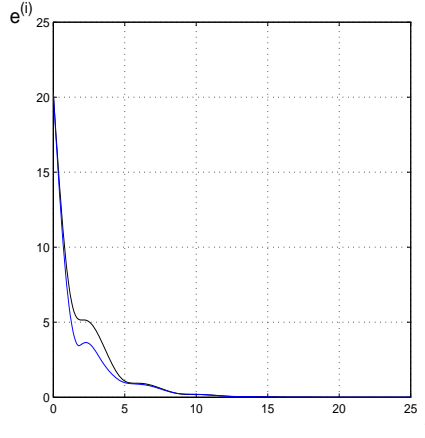


Figure 3: Performance with identical ($e^{(1)}(t)$; upper line) and nonidentical ($e^{(2)}(t)$; lower line) gains.

References

- [1] R. Agaev and P. Chebotarev. The matrix of maximum out forests of a digraph and its applications. *Automation and Remote Control*, 61(9):1424–1450, 2000.
- [2] R. Bellman. *Introduction to Matrix Analysis*. McGraw-Hill, Inc., 1960.
- [3] S. Boyd, L. Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in System and Control Theory*, volume 15 of *Studies in Applied Mathematics*. SIAM, 1994.
- [4] F. Bullo, J. Cortez, and S. Martinez. *Distributed control of robotic networks*. Princeton Univ. Press, 2009.
- [5] P. Chebotarev and R. Agaev. The forest consensus theorem and asymptotic properties of coordination protocols. *4th IFAC Workshop on Distributed Estimation and Control in Networked Systems, Germany*, pages 95–101, 2013.
- [6] J. R. Fax and R. M. Murray. Information flow and cooperative control of vehicle formations. *IEEE Trans. on Autom. Control*, 49:1465–1476, 2004.
- [7] A. Fradkov and I. Junussov. Synchronization of linear object networks by output feedback. *50th IEEE Conf. on Decision and Control and European Control Conf. (CDC-ECC)*, pages 8188 – 8192, 2011.
- [8] A. L. Fradkov. Quadratic lyapunov functions in the adaptive stabilization problem of a linear dynamic plant. *Siberian Math J*, 2:341–348, 1976.
- [9] A. L. Fradkov. Passification of nonsquare linear systems and feedback yakubovich-kalman-popov lemma. *Eur. J. Control*, (6):573–582, 2003.
- [10] I. Gutman and D. Vidovic. The largest eigenvalues of adjacency and laplacian matrices, and ionization potentials of alkanes. *Indian J. Chem.*, 41A:893–896, 2002.
- [11] R. Horn and C. Johnson. *Matrix analysis*. Cambridge University Press, 1990.
- [12] Z. K. Li, Z. S. Duan, G. R. Chen, and L. Huang. Consensus of multi-agent systems and synchronization of complex networks: A unified viewpoint. *IEEE Trans. On Circuits And Systems-I: Reg. Papers*, 57(1):213–224, 2010.
- [13] M. Marcus and H. Minc. *A Survey of Matrix Theory and Matrix Inequalities*. Allyn and Bacon, Inc., Boston, 1964.

- [14] R. Olfati-Saber, J. Fax, and R. Murray. Consensus and cooperation in networked multi-agent systems. *Proc. IEEE*, 95(1):215–233, January 2007.
- [15] R. Olfati-Saber and R. M. Murray. Consensus problems in networks of agents with switching topology and time-delays. *IEEE Trans. on Autom. Control*, 49(9):1520–1533, 2004.
- [16] A. A. Peters, R. H. Middleton, and O. Mason. Leader tracking in homogeneous vehicle platoons with broadcast delays. *Automatica*, 2013.
- [17] W. Ren, R. W. Beard, and E. M. Atkins. A survey of consensus problems in multi-agent coordination. *Proc. of American Control Conference, Oregon*, pages 1859–1864, 2005.
- [18] C. Reynolds. Flocks, herds, and schools: A distributed behavioral model. *Computer Graphics*, 21(4):25–34, July 1987.
- [19] P. Seiler, A. Pant, and K. Hedrick. Disturbance propagation in vehicle strings. *IEEE Trans. on Autom. Control*, 49(10):1835–1841, October 2004.
- [20] F. M. Tagerman, J. P. Veerman, and B. D. Stosic. Asymmetric decentralized flocks. *IEEE Trans. on Autom. Control*, 57(11):2844–2853, 2012.
- [21] W. Yu, W. Ren, W. Zheng, G. Chen, and J. Lu. Distributed control gains design for consensus in multi-agent systems with second-order nonlinear dynamics. *Automatica*, 49:2107–2115, 2013.
- [22] D. Zelazo and M. Mesbahi. Edge agreement: Graph theoretic performance bounds and passivity analysis. *IEEE Trans. on Autom. Control*, 56(3):544–555, 2011.
- [23] H. Zhang, F. L. Lewis, and A. Das. Optimal design for synchronization of cooperative systems: State feedback, observer and output feedback. *IEEE Trans. on Autom. Control*, 56(8):1948–1952, August 2011.